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Combinations of multivariate averages

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Abstract

Rate of approximation of combinations of averages on the spheres is shown to be equivalent to K-functionals yielding higher degree of smoothness. Results relating combinations of averages on rims of caps of spheres are also achieved.

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1. Introduction

In a recent paper [Be-Da-Di] the average on a sphere of radius t in R^d, $d \geqslant 2$ given by

$$V_t f(x) = \frac{1}{m(t)} \int_{\{y \in R^d : |x-y|=t\}} f(y)\, d\sigma(y), \quad V_t 1 = 1, \quad x \in R^d \tag{1.1}$$

(where $d\sigma(y)$ is a measure invariant under rotations about x) was shown to satisfy an equivalence relation with the appropriate K-functionals, that is

$$\|V_t f - f\|_{L_p(R^d)} \approx \inf\left(\|f-g\|_{L_p(R^d)} + t^2\|\Delta g\|_{L_p(R^d)}\right) \equiv K(f, \Delta, t^2)_p, \tag{1.2}$$

where $1 \leqslant p \leqslant \infty$, $d \geqslant 2$ and Δ is the Laplacian i.e. $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$.

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The average on the rim of the cap of the sphere

$$S^{d-1} = \{x \in R^d : |x|^2 = x_1^2 + \cdots + x_d^2 = 1\}$$

given by

$$S_\theta f(x) = \frac{1}{m(\theta)} \int_{\{y \in S^{d-1}: x \cdot y = \cos\theta\}} f(y)\, d\gamma(y), \quad S_\theta 1 = 1, \quad x \in S^{d-1} \tag{1.3}$$

(where $d\gamma(y)$ is a measure on the set $\{y \in S^{d-1} : x \cdot y = \cos\theta\}$ invariant under rotation about x) was shown in [Be-Da-Di] to satisfy the equivalence relation

$$\begin{aligned} \|S_\theta f - f\|_{L_p(S^{d-1})} &\approx \inf \left(\|f - g\|_{L_p(S^{d-1})} + \theta^2 \|\widetilde{\Delta} g\|_{L_p(S^{d-1})} \right) \\ &\equiv K(f, \widetilde{\Delta}, \theta^2)_p, \end{aligned} \tag{1.4}$$

where $1 \leqslant p \leqslant \infty$, $d \geqslant 3$ and $\widetilde{\Delta}$ is the Laplace–Beltrami operator given by $\widetilde{\Delta} f(x) = \Delta f\left(\frac{x}{|x|}\right)$ for $x \in S^{d-1}$.

We will show here that

$$V_{\ell,t} f(x) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} V_{jt} f(x) \tag{1.5}$$

satisfies for $d \geqslant 2$ and $1 \leqslant p \leqslant \infty$

$$\begin{aligned} \|V_{\ell,t} f(\cdot) - f(\cdot)\|_{L_p(R^d)} &\approx \inf_g \left(\|f - g\|_{L_p(R^d)} + t^{2\ell} \|\Delta^\ell g\|_{L_p(R^d)} \right) \\ &\equiv K_\ell(f, \Delta, t^{2\ell})_p, \end{aligned} \tag{1.6}$$

where $\Delta^\ell g = \Delta(\Delta^{\ell-1} g)$.

We will also show that

$$S_{\ell,\theta} f(x) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} S_{j\theta} f(x) \tag{1.7}$$

satisfies

$$\begin{aligned} \|S_{\ell,\theta} f(\cdot) - f(\cdot)\|_{L_p(S^{d-1})} &\approx \inf_g \left(\|f - g\|_{L_p(S^{d-1})} + \theta^{2\ell} \|\widetilde{\Delta}^\ell g\|_{L_p(S^{d-1})} \right) \\ &\equiv K_\ell(f, \widetilde{\Delta}, \theta^{2\ell})_p, \end{aligned} \tag{1.8}$$

where $\widetilde{\Delta}^{\ell} g = \widetilde{\Delta}(\widetilde{\Delta}^{\ell-1} g)$.

The main thrust of this paper is that in both (1.6) and (1.8) there is no supremum sign on the left-hand side as was the case in previous results on combinations (see for instance [Li-Ni, Ni-Li, Ru]). One should note that only ℓ elements are needed to achieve K-functionals whose saturation rate is $O(t^{2\ell})$ (or $O(\theta^{2\ell})$).

2. Realization, Bernstein and Jackson results on R^d

To prove (1.6) we need some preliminary results that we hope will be useful elsewhere as well. Given $\eta(y) \in C^\infty(R_+)$, $\eta(y) = 1$ for $y \leqslant 1$ and $\eta(y) = 0$ for $y \geqslant 2$, we define $\eta_R(f)$ by

$$\big(\eta_R(f)\big)^\wedge(x) = \eta\Big(\frac{|x|}{R}\Big)\widehat{f}(x), \quad R > 0, \tag{2.1}$$

where

$$\widehat{g}(x) = \int_{R^d} g(\xi)e^{-2\pi i \xi\cdot x}\, d\xi. \tag{2.2}$$

In what follows we will use extensively the basic properties of the multivariate Fourier transform which are given for instance in the first two chapters of Stein and Weiss [St-We].

Setting

$$G(x) = \int_{R^d} \eta(t)e^{2\pi i t x}\, dt$$

and following Lemma 3.17 of Stein and Weiss [St-We, p. 26], we have $G \in L_1(R^d)$. Hence, using [St-We, (1.6), p. 4], it is clear that there exists $G_R(x) \in L_1(R^d)$ such that

$$\eta_R(f)(x) = G_R * f(x) \quad \text{for} \quad f \in L_p(R^d), \tag{2.3}$$

$$G_R(x) = R^d G(Rx), \quad G(x) = G_1(x) \tag{2.4}$$

and

$$\|G_R\|_{L_1} = \|G\|_{L_1}. \tag{2.5}$$

The Bernstein-type inequality is given in the following result.

Theorem 2.1. *Suppose $f \in L_p(R^d)$, $1 \leqslant p \leqslant \infty$ and* supp $\widehat{f} \subset \{|x| : |x| \leqslant R\}$. *Then $\Delta^\ell f$ exists in L_p and*

$$\|\Delta^\ell f\|_p \leqslant C R^{2\ell}\|f\|_p \tag{2.6}$$

with C independent of R and p.

Proof. We note first that when we described $\widehat{f}$ and its support, we did not imply that it is a function, and in fact for $2 < p \leqslant \infty$ it may be just an element of $\mathcal{S}'$ (the dual to $\mathcal{S}$). However, G_R given in (2.1) and (2.3) is in L_1, and using [St-We, (1.9), p. 5] on $\eta(t)$ and $G(x)$, and following the argument yielding $G_R \in L_1$, so is $\Delta^\ell G_R(x)$ where Δ is the Laplacian. Moreover,

$$\left(-4\pi^2\left(\frac{|x|}{R}\right)^2\right)^\ell \eta\left(\frac{|x|}{R}\right) = \frac{1}{R^{2\ell}}\big(\Delta^\ell G_R\big)^\wedge(x)$$

and hence

$$\frac{1}{R^{2\ell}}\,\|\Delta^\ell G_R\|_{L_1} = \|\Delta^\ell G\|_{L_1} = A(\ell).$$

This implies for $\ell = 0, 1, \ldots$

$$\frac{1}{R^{2\ell}}\,\|\Delta^\ell \eta_R f\|_{L_p} \leqslant A(\ell)\|f\|_{L_p}.$$

If $f \in L_p$ such that supp $\widehat{f} \subset \{|x| : |x| \leqslant R\}$, $\eta_R(f) = f$, and hence $\Delta^\ell(\eta_R f) = \Delta^\ell f$, and (2.6) is satisfied with $C = A(\ell)$. $\square$

For $f \in L_p(R^d)$ we define the rate of best approximation by

$$E_\lambda(f)_p = \inf\ \big\{\|f - h_\lambda\|_p : h_\lambda \in L_p(R^d),\ \text{supp}\ \big(h_\lambda{}^\wedge(x)\big) \subset B_\lambda\big\}, \tag{2.7}$$

where $B_\lambda \equiv \{x : |x| \leqslant \lambda\}$.

We can now state and prove the Jackson-type result.

Theorem 2.2. *For $f \in L_p(R^d)$, $1 \leqslant p \leqslant \infty$, we have*

$$E_\lambda(f)_p \leqslant \inf_g\,(\|f - g\|_p + \lambda^{-2\ell}\|\Delta^\ell g\|_p) \equiv K_\ell(f, \Delta, \lambda^{-2\ell})_p. \tag{2.8}$$

Proof. We define $\mathcal{R}_{\lambda,\ell,b}(f)$ for $\ell = 1, 2, \ldots,$ and $b \geqslant d + 2$ by

$$\big(\mathcal{R}_{\lambda,\ell,b}(f)\big)^\wedge(x) = \begin{cases} \Big(1 - \big(\frac{|x|}{\lambda}\big)^{2\ell}\Big)^b \widehat{f}(x) & |x| \leqslant \lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

We note that while $b \geqslant d + 2$ may not be necessary, it is convenient. (Using $\mathcal{R}_{\lambda,\ell,b}(f)$ is also just for convenience.) The function

$$\Phi_{\ell,b}(x) = \begin{cases} (1 - |x|^{2\ell})^b, & |x| \leqslant 1, \\ 0 & \text{otherwise} \end{cases}$$

satisfies $\|D^\nu \Phi_{\ell,b}\|_{L_1} \leqslant C(\ell, b)$ for $|\nu| \leqslant d + 1$, and hence there exists $G_{\ell,b}{}^\wedge(x) = \Phi_{\ell,b}(x)$ such that $G_{\ell,b}(\xi) \in L_1(R^d)$, $\int_{R^d} G_{\ell,b}(\xi)\,d\xi = 1$, and moreover $G_{\ell,b}(\xi) = G_{\ell,b}(\rho\xi)$ for any orthogonal matrix ρ with determinant 1, $\rho \in SO(d)$. We now have

$$\mathcal{R}_{\lambda,\ell,b}(f)(\xi) = \lambda^d \int_{R^d} G_{\ell,b}\big(\lambda(\xi - \eta)\big) f(\eta)\,d\eta. \tag{2.10}$$

We recall the definition of $K_\ell(f, \Delta, \lambda^{-2\ell})_p$ and choose g_1 such that

$$\|f - g_1\|_p + \lambda^{-2\ell}\|\Delta^\ell g_1\|_p \leqslant 2K_\ell(f, \Delta, \lambda^{-2\ell})_p.$$

Using (2.10), we have

$$\|\mathcal{R}_{\lambda,\ell,b}(f - g_1) - (f - g_1)\|_p \leqslant (C + 1)\|f - g_1\|_p \leqslant (C + 1)2K_\ell(f, \Delta, \lambda^{-2\ell})_p.$$

To estimate $\mathcal{R}_{\lambda,\ell,b}(g_1) - g_1$, we write

$$\begin{aligned}\|\mathcal{R}_{\lambda,\ell,b}(g_1) - g_1\|_p &\leqslant \|\mathcal{R}_{\lambda,\ell,b}(g_1) - \mathcal{R}_{\lambda,\ell,b+1}(g_1)\|_p \\ &\quad + \|\mathcal{R}_{\lambda,\ell,b+1}(g_1) - \mathcal{R}_{\Lambda,\ell,b+1}(g_1)\|_p + \|\mathcal{R}_{\Lambda,\ell,b+1}(g_1) - g_1\|_p \\ &\equiv I_1(\lambda)_p + I_2(\lambda,\Lambda)_p + I_3(\Lambda)_p.\end{aligned}$$

For $g_1 \in L_p(R^d)$, $1 \leqslant p < \infty$, $I_3(\Lambda)_p \to 0$ as $\Lambda \to \infty$. For $p = \infty$ if $\Delta^\ell g_1 \in L_\infty$, $g_1 \in C_0(R^d)$, and hence $I_3(\Lambda)_p \to 0$ as $\Lambda \to \infty$. To estimate $I_1(\lambda)_p$ we write

$$\begin{aligned}\mathcal{R}_{\lambda,\ell,b}(g_1) - \mathcal{R}_{\lambda,\ell,b+1}(g_1) &= \frac{1}{\lambda^{2\ell}} \frac{1}{(-4\pi^2)^\ell} \Delta^\ell\big(\mathcal{R}_{\lambda,\ell,b}(g_1)\big) \\ &= \frac{1}{\lambda^{2\ell}} \frac{1}{(-4\pi^2)^\ell} \mathcal{R}_{\lambda,\ell,b}(\Delta^\ell g_1)\end{aligned}$$

and hence

$$I_1(\lambda)_p \leqslant \frac{C}{\lambda^{2\ell}} \|\Delta^\ell g_1\|_p \leqslant C_1 K_\ell(f,\Delta,\lambda^{-2\ell})_p.$$

To estimate $I_2(\lambda,\Lambda)_p$ we write

$$\mathcal{R}_{\lambda,\ell,b+1}(g_1) - \mathcal{R}_{\Lambda,\ell,b+1}(g_1) = \frac{(b+1)2\ell}{(-4\pi^2)^\ell} \int_\lambda^\Lambda \Delta^\ell \mathcal{R}_{\mu,\ell,b}(g_1) \frac{d\mu}{\mu^{2\ell+1}}$$

and as

$$\begin{aligned}\|\Delta^\ell \mathcal{R}_{\mu,\ell,b}(g_1)\|_p &= \|\mathcal{R}_{\mu,\ell,b}(\Delta^\ell g_1)\|_p \\ &\leqslant C\|\Delta^\ell g_1\|_p,\end{aligned}$$

we have

$$I_2(\lambda,\Lambda)_p \leqslant \frac{C(b+1)}{(4\pi^2)^\ell} \frac{1}{\lambda^{2\ell}} \|\Delta^\ell g_1\|_p \leqslant C_2 K_\ell(f,\Delta,\lambda^{-2\ell})_p.$$

This implies

$$\|f - \mathcal{R}_{\lambda,\ell,b}(f)\|_p \leqslant C_3 K_\ell(f,\Delta,\lambda^{-2\ell})_p \tag{2.11}$$

and hence (2.8). □

Corollary 2.3. *For $f \in L_p(R^d)$, $1 \leqslant p \leqslant \infty$, $\lambda \geqslant 0$*

$$K_\ell(f,\Delta,\lambda^{-2\ell})_p \approx \|f - \mathcal{R}_{\lambda,\ell,b}(f)\|_p + \lambda^{-2\ell}\|\Delta^\ell \mathcal{R}_{\lambda,\ell,b}(f)\|_p. \tag{2.12}$$

Proof. By definition the left-hand side is bounded by the right-hand side. Using (2.11), we have to show only that $\lambda^{-2\ell}\|\Delta^\ell \mathcal{R}_{\lambda,\ell,b}(f)\|_p$ is bounded by the left-hand side. We recall that

$$\frac{1}{\lambda^{2\ell}} \Delta^\ell \mathcal{R}_{\lambda,\ell,b}(f) = (-4\pi^2)^\ell(\mathcal{R}_{\lambda,\ell,b} f - \mathcal{R}_{\lambda,\ell,b+1} f)$$

and we complete the proof observing that

$$\|\mathcal{R}_{\lambda,\ell,b}(f) - \mathcal{R}_{\lambda,\ell,b+1}(f)\|_p \leqslant \|\mathcal{R}_{\lambda,\ell,b}(f) - f\|_p + \|f - \mathcal{R}_{\lambda,\ell,b+1}(f)\|_p,$$

which, using (2.11) for b and $b+1$, yields our result. □

Corollary 2.4. *Suppose $\eta_\lambda(f)$ is defined by* (2.1) *and $\mathcal{R}_{\lambda,\ell,b}(f)$ is given by* (2.9) *with $b \geqslant d+2$, then*

$$\|f-\eta_\lambda(f)\|_p \leqslant C\|f-\mathcal{R}_{\lambda,\ell,b}(f)\|_p. \tag{2.13}$$

Proof. Using $\eta_\lambda\big(\mathcal{R}_{\lambda,\ell,b}(f)\big)=\mathcal{R}_{\lambda,\ell,b}(f)$, we write

$$\begin{aligned}\|f-\eta_\lambda(f)\|_p &= \|f-\mathcal{R}_{\lambda,\ell,b}(f)-\eta_\lambda\big(f-\mathcal{R}_{\lambda,\ell,b}(f)\big)\|_p\\ &\leqslant (1+\|G\|_1)\,\|f-\mathcal{R}_{\lambda,\ell,b}(f)\|_p\end{aligned}$$

since $\|\eta_\lambda(f)\|_p \leqslant \|G\|_1\|f\|_p$. This is, in fact, the routine de la Valleé Poussin procedure. □

Corollary 2.5. *For $\eta_\lambda(f)$ given by* (2.1)

$$K_\ell(f,\Delta,\lambda^{-2\ell})_p \approx \|f-\eta_\lambda(f)\|_p+\lambda^{-2\ell}\|\Delta^\ell\eta_\lambda(f)\|_p. \tag{2.14}$$

Proof. Using the definition of $K_\ell(f,\Delta,\lambda^{-2\ell})_p$, the inequality (2.13) and the equivalence (2.12), we have to estimate only

$$\begin{aligned}\lambda^{-2\ell}\|\Delta^\ell\eta_\lambda f\|_p &\leqslant \lambda^{-2\ell}\|\Delta^\ell\mathcal{R}_{\lambda,\ell,b}(f)\|_p+\lambda^{-2\ell}\|\Delta^\ell\big(\eta_\lambda(f)-\mathcal{R}_{\lambda,\ell,b}(f)\big)\|_p\\ &\leqslant C K_\ell(f,\Delta,\lambda^{-2\ell})_p+\lambda^{-2\ell}C_1\lambda^{2\ell}\|\eta_\lambda(f)-\mathcal{R}_{\lambda,\ell,b}(f)\|_p\\ &\leqslant C_2K_\ell(f,\Delta,\lambda^{-2\ell})_p. \qquad \square\end{aligned}$$

3. Strong converse inequality on R^d

The main result of this section is the equivalence (1.6) given in the following theorem:

Theorem 3.1. *For $d>1$, $\ell=1,2,\ldots,$ $t>0$, $V_{t,\ell}f$ given by* (1.5) *and $1\leqslant p\leqslant\infty$ we have*

$$\|V_{\ell,t}f-f\|_{L_p(R^d)} \approx \inf_g \Big(\|f-g\|_{L_p}+t^{2\ell}\|\Delta^\ell g\|_{L_p}\Big). \tag{3.1}$$

For the proof we will need several lemmas.

Lemma 3.2. *For an integer ℓ we have*

$$\binom{2\ell}{\ell}+2\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell-j}\cos\, j\theta = 4^\ell\,\sin^{2\ell}\frac{\theta}{2}. \tag{3.2}$$

Proof. Writing $\cos\, j\theta=\frac{1}{2}\,(e^{ij\theta}+e^{-ij\theta})$ and $\sin\frac{\theta}{2}=\frac{1}{2i}\,(e^{i\theta/2}-e^{-i\theta/2})$, we obtain (3.2) by simple computation. □

Lemma 3.3. *For $V_{\ell,t}(f)$ given in* (1.5)

$$(V_{\ell,t}f)^\wedge(x)\equiv m_\ell(2\pi t|x|)\widehat{f}(x) \tag{3.3}$$

and

$$1 - m_\ell(u) = \frac{2\Gamma(\frac{d}{2})}{\Gamma(\frac{d-1}{2})\Gamma(\frac{1}{2})} \frac{4^\ell}{\binom{2\ell}{\ell}} \int_0^1 \big(\sin \frac{us}{2}\big)^{2\ell} (1-s^2)^{\frac{d-3}{2}}\, ds. \tag{3.4}$$

Proof. It is known that

$$(V_t f)^\wedge(x) = m_1(2\pi t|x|)\widehat{f}(x) = m(2\pi t|x|)\widehat{f}(x)$$

with (see [St-We, pp. 153–154])

$$\begin{aligned} m(u) &= \Gamma\Big(\frac{d}{2}\Big)\Big(\frac{u}{2}\Big)^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(u) \\ &= \frac{2\Gamma\big(\frac{d}{2}\big)}{\Gamma\big(\frac{d-1}{2}\big)\Gamma\big(\frac{1}{2}\big)} \int_0^1 \cos us\ (1-s^2)^{\frac{d-3}{2}}\, ds, \end{aligned}$$

where $J_{\frac{d-2}{2}}(u)$ is the Bessel function given by the above formula. We now use the definition of $V_{\ell,t}(f)$ to obtain

$$\begin{aligned} m_\ell(u) &= \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} m(ju) \\ &= \frac{2\Gamma\big(\frac{d}{2}\big)}{\Gamma\big(\frac{d-1}{2}\big)\Gamma\big(\frac{1}{2}\big)} \int_0^1 \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} \cos\ jus\ (1-s^2)^{\frac{d-3}{2}}\, ds. \end{aligned}$$

Using Lemma 3.2, we now derive (3.4). □

Lemma 3.4. *For* $0 < u \leqslant \pi$

$$0 < C_1 u^{2\ell} \leqslant 1 - m_\ell(u) \leqslant C_2 u^{2\ell}. \tag{3.5}$$

For $u \geqslant \pi$

$$0 < m_\ell(u) \leqslant v_{d,\ell} < 1. \tag{3.6}$$

Proof. For $0 < \frac{us}{2} < \frac{\pi}{2}$ ($u < \pi$, $0 \leqslant s \leqslant 1$) we have $(\frac{us}{\pi})^2 \leqslant \sin^2 \frac{us}{2} \leqslant (\frac{us}{2})^2$, which, using (3.4), implies (3.5) (with C_1 and C_2 depending on d and ℓ). For $u \geqslant \pi$

$$\begin{aligned} 1 - m_\ell(u) &\geqslant \frac{2\big(\frac{d}{2}\big)}{\Gamma\big(\frac{d-1}{2}\big)\Gamma\big(\frac{1}{2}\big)} \frac{4^\ell}{\binom{2\ell}{\ell}} \int_0^{2/3} \Big(\sin \frac{us}{2}\Big)^{2\ell} (1-s^2)^{\frac{d-3}{2}}\, ds \\ &\geqslant \frac{2\Gamma\big(\frac{d}{2}\big) 4^\ell \big(\frac{1}{2}\big)^{\frac{d-3}{2}}}{\Gamma\big(\frac{d-1}{2}\big)\Gamma\big(\frac{1}{2}\big)\binom{2\ell}{\ell}} \int_0^{2/3} \Big(\sin \frac{us}{2}\Big)^{2\ell} ds \\ &\equiv C_{d,\ell} \int_0^{2/3} \Big(\sin \frac{us}{2}\Big)^{2\ell} ds \\ &= C_{d,\ell}\, \frac{1}{u} \int_0^{\frac{2}{3}u} \Big(\sin \frac{\zeta}{2}\Big)^{2\ell} d\zeta \end{aligned}$$

$$\geqslant C_{d,\ell}\,\frac{1}{u}\sum_{b=1}^{[u/\pi]}\int_{\pi/3}^{2\pi/3}\left(\sin\,\frac{\zeta}{2}\right)^{2\ell}d\zeta$$
$$= C_{d,\ell}\,\frac{1}{u}\left[\frac{u}{\pi}\right]\left(\frac{1}{2}\right)^{2\ell}\frac{\pi}{3}\geqslant C_{d,\ell}>0. \qquad \square$$

Lemma 3.5. *For $j=0,1,2,\ldots,$ and $u\geqslant 0$*

$$\left|\left(\frac{d}{du}\right)^{j}m_\ell(u)\right|\leqslant C_{\ell,j}\left(\frac{1}{1+u}\right)^{\frac{d-1}{2}}. \tag{3.7}$$

Proof. As $m_\ell(u)$ is a linear combination of $m(ku)$, $1\leqslant k\leqslant\ell$, it is sufficient to prove (3.7) for $\ell=1$. Recalling the definition of $J_k(t)$ [St-We, p. 153], $\frac{d}{dt}\big(t^{-k}J_k(t)\big)=-t^{-k}J_{k+1}(t)$ and [St-We, Lemma 3.11, p. 158], we have our result. $\square$

Proof of Theorem 3.1. Using Corollary 2.5 and the definition of the K-functional $K_\ell(f,\Delta,t^{2\ell})_p$, we have only to show for all $f\in L_p(R^d)$ and some fixed $a>0$ (as $K_\ell(f,\Delta,t^{2\ell})_p \approx K_\ell(f,\Delta,a^{-2\ell}t^{2\ell})_p$) that

$$\|f-V_{\ell,t}f\|_p\geqslant C_1\|f-\eta_{a/t}f\|_p, \tag{3.8}$$

$$\|f-V_{\ell,t}f\|_p\geqslant C_2t^{2\ell}\|\Delta^\ell\eta_{a/t}f\|_p, \tag{3.9}$$

and

$$\|\eta_{a/t}(f)-V_{\ell,t}\eta_{a/t}(f)\|_p\leqslant C_3t^{2\ell}\|\Delta^\ell\eta_{a/t}(f)\|_p. \tag{3.10}$$

To prove (3.8) it is sufficient to show

$$\begin{aligned}&\|(I-\eta_{a/t})f-(I-\eta_{a/t})(I+V_{\ell,t}+V_{\ell,t}^2+V_{\ell,t}^3+V_{\ell,t}^4)(f-V_{\ell,t}f)\|_p\\&\quad\leqslant C_4\|f-V_{\ell,t}f\|_p\end{aligned} \tag{3.11}$$

since, as $\eta_{1/t}$ and $V_{\ell,t}$ are bounded multiplier operators on $L_p(R^d)$, we have

$$\|(I-\eta_{a/t})(I+V_{\ell,t}+V_{\ell,t}^2+V_{\ell,t}^3+V_{\ell,t}^4)(f-V_{\ell,t}f)\|_p\leqslant C_5\,\|f-V_{\ell,t}f\|_p,$$

where I is the identity operator. To prove (3.11) we have to show that

$$\Phi(u)=\frac{\big(1-\eta(u/a)\big)m_\ell(u)^5}{1-m_\ell(u)}$$

is a bounded multiplier on $L_1(R^d)$ (and hence on $L_p(R^d)$), or $|D^\nu\Phi(u)|\leqslant\frac{C}{(1+|u|)^{d+\alpha}}$, $\alpha>0$ (at least for $|\nu|\leqslant d+1$, but here that restriction does not matter). While the above is known and used numerous times, we show it below to help the reader. For $\overset{\vee}{\Phi}(x)$ given by

$$\overset{\vee}{\Phi}(x)=\int_{R^d}\Phi(y)e^{2\pi ixy}\,dy,$$

which may be considered as a Fourier transform, and following the proof of Lemma 3.17 of [St-We, p. 26], we have

$$\| \overset{\vee}{\Phi} \|_{L_1(R^1)} \leqslant C \sum_{|\alpha| \leqslant d+1} \| D^\alpha \Phi \|_{L_1(R^d)},$$

which implies the sufficiency of showing that $|D^\nu \Phi(u)| \leqslant \frac{C}{(1+|u|)^{d+\alpha}}$ for $\alpha > 0$ and $|\nu| \leqslant d+1$. We note that for $|u| \leqslant 1$, $\Phi(u) = 0$. For $|u| \geqslant 1$ we use Lemma 3.5, recall that the multipliers we have are radial, and obtain

$$|D^\nu \Phi(u)| \leqslant C(\nu) \Big(\frac{1}{1+|u|} \Big)^{5\left(\frac{d-1}{2}\right)} = C(\nu) \Big(\frac{1}{1+|u|} \Big)^{d+\frac{3}{2}d-\frac{5}{2}}$$

and for $d \geqslant 2$ we have $\frac{3d}{2} - \frac{5}{2} \geqslant 3 - \frac{5}{2} = \frac{1}{2} > 0$.

To prove (3.9) we have to show that

$$\Psi(u) = \frac{u^{2\ell} \eta\left(\frac{u}{a}\right)}{1 - m_\ell(u)}$$

is a multiplier. As $\eta\left(\frac{u}{a}\right) = 0$ for $|u| > 2a$, we just have to check that $\frac{u^{2\ell}}{1-m_\ell(u)}$ and its derivatives are bounded for $|u| \leqslant 2a$. The boundedness of $\frac{u^{2\ell}}{1-m_\ell(u)}$ follows from (3.5) of Lemma 3.4 (for $a \leqslant \frac{\pi}{2}$) as C_1 there satisfies $C_1 > 0$. We follow Lemma 3.3 to observe that $1 - m_\ell(z)$ given by

$$1 - m_\ell(z) = \frac{2\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \frac{4^{2\ell}}{\binom{2\ell}{\ell}} \int_0^1 \left(\sin \frac{zs}{2} \right)^{2\ell} (1-s^2)^{\frac{d-3}{2}} \, ds$$

is an analytic function which, using Lemma 3.4, has a zero of order 2ℓ at 0. As 0 is an isolated zero, $1 - m_\ell(z) \neq 0$ for $0 < |z| \leqslant 2a$ for some a and hence $\frac{z^{2\ell}}{1-m_\ell(z)}$ is analytic there, and therefore $\Psi(u)$ is in $C^\infty[0, \infty)$ as required. To estimate (3.10) we have to show that

$$\Psi_1(u) = \frac{1 - m_\ell(u)}{u^{2\ell}} \, \eta\Big(\frac{u}{a}\Big)$$

is a multiplier. For this we use the fact that in (3.5) of Lemma 3.4 $C_2 < \infty$ and $m_\ell(u)\eta\left(\frac{u}{a}\right) \in C^\infty[0, \infty)$ as proved earlier. $\square$

4. Combinations of averages on the sphere

Our goal is to prove the equivalence (1.8) for functions on the sphere. This result is summarized in the following theorem.

Theorem 4.1. *For $f \in L_p(S^{d-1})$, $d \geqslant 3$, $1 \leqslant p \leqslant \infty$, $\ell = 1, 2, \ldots$, and $0 < \theta \leqslant \frac{\pi}{2\ell}$ we have*

$$\begin{aligned} \|S_{\theta,\ell} f - f\|_p &\approx \inf \left(\|f - g\|_p + \theta^{2\ell} \|\widetilde{\Delta}^{\ell} g\|_p \right) \\ &\equiv K_\ell(f, \widetilde{\Delta}, \theta^{2\ell})_p, \end{aligned} \tag{4.1}$$

where $S_{\theta,\ell} f$ is given by (1.7) *and $\widetilde{\Delta}$ is the Laplace–Beltrami operator.*

We cannot expect (4.1) for all t as $S_\theta f = S_{2\pi-\theta} f$, and for $\ell = 1$ this would imply $K_1(f, \widetilde{\Delta}, \theta^2)_p \approx K_1\left(f, \widetilde{\Delta}, (2\pi-\theta)^2\right)_p$, and hence $K_1\left(f, \widetilde{\Delta}, (2\pi-\theta)^2\right)_p \leqslant C K_1(f, \widetilde{\Delta}, \theta^2)_p$, which if C is independent of θ, is valid only for $f = \text{const}$. We will prove Theorem 4.1 in Section 5, and this section is dedicated to the numerous lemmas needed for that proof.

Lemma 4.2. *The operator $S_{\theta,\ell} f$ is a bounded multiplier operator*

$$S_{\theta,\ell} f(x) = \sum_{k=0}^{\infty} a_\ell(k, \theta) P_k f, \tag{4.2}$$

where $P_k f$ is the projection on $H_k = \{\Psi : \widetilde{\Delta}\Psi = -k(k+d-2)\Psi\}$, and $a_\ell(k, \theta)$ is given by

$$a_\ell(k, \theta) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} Q_k^\lambda(\cos j\theta), \tag{4.3}$$

where $Q_k^\lambda(t)$ are the ultraspherical polynomials with $\lambda = \frac{d-2}{2}$ normalized by $Q_k^\lambda(1) = 1$.

Proof. The above is just a compilation of the known facts on $S_\theta f$ substituted in the definition of $S_{\theta,\ell} f$. (One may consult [Be-Da-Di] for details on $P_k(S_\theta f)$ and other details.) $\square$

Lemma 4.3. *For $a_\ell(k, \theta)$ given by* (4.3) *and $0 < \theta \leqslant \frac{\pi}{2\ell}$ we have*

$$|\Delta^j a_\ell(k, \theta)| \leqslant \begin{cases} C\, \theta^j & \text{if } 0 < k\theta \leqslant 1, \\ C\theta^j \left(\frac{1}{k\theta}\right)^\lambda & \text{if } k\theta \geqslant 1, \end{cases}$$

where $\Delta^0 b_k = b_k$, $\Delta b_k = b_{k+1} - b_k$, $\Delta^j b_k = \Delta(\Delta^{j-1} b_k)$, $\lambda = \frac{d-2}{2}$ and j is an integer $j \geqslant 0$.

Proof. Using (4.3), we may apply [Be-Da-Di, Lemma 3.2] with $m = 1$ and $r\theta$ for θ with $r = 1, \ldots, \ell$ to obtain

$$|\Delta^j Q_k^{(\lambda)}(\cos r\theta)| \leqslant \begin{cases} C\, \theta^j/(k\theta)^\lambda & \text{for } kr\theta \geqslant 1, \\ C\, \theta^j & \text{for } kr\theta \leqslant 1, \end{cases} \tag{4.4}$$

from which (4.4) follows when we recall that for $k\theta \approx 1$, the difference between the two estimates can be inserted in the constant. For $j = 0$ (4.4) is contained in [Sz, (7.33.6), 170]. $\square$

Lemma 4.4. *For $a_\ell(k,\theta)$ given by* (4.3) *and* $\theta \in [0, \frac{\pi}{2}]$ *we have*

$$0 < C_1 \leqslant \frac{1 - a_\ell(k,\theta)}{(k\theta)^{2\ell}} \leqslant C_2 < \infty \quad \textit{for} \quad 0 < k\theta \leqslant \pi \tag{4.5}$$

and for any $\tau > 0$

$$a_\ell(k,\theta) \leqslant v_{d,\ell,\tau} < 1 \quad \textit{for} \quad k\theta \geqslant \tau > 0. \tag{4.6}$$

Proof. We use [Sz, (4.9.19), p. 95] to write

$$Q_k^{(\lambda)}(\cos\,\theta) = \sum_{v=0}^{[k/2]} \alpha(k, 2v, \lambda) \cos\,(k-2v)\theta, \tag{4.7}$$

where (using [Sz, (4.9.21) and (4.7.3)])

$$\alpha(k, 2v, \lambda) = \frac{2\binom{k-v+\lambda-1}{k-v}\binom{v+\lambda-1}{v}}{\binom{k+2\lambda-1}{k}}\ . \tag{4.8}$$

Using (4.3) and (3.2), we have

$$1 - a_\ell(k,\theta) = \frac{4^\ell}{\binom{2\ell}{\ell}} \sum_{v=0}^{[k/2]} \alpha(k, 2v, \lambda) \sin^{2\ell}\, \frac{k-2v}{2}\, \theta. \tag{4.9}$$

For $k\theta \leqslant \pi$ we recall that $\sum_{v=0}^{[k/2]} \alpha(k, 2v, \lambda) = 1$ (setting $\theta = 0$ in (4.7)) and that $\sin^{2\ell}\, \frac{k-v}{2}\,\theta \leqslant \sin^{2\ell}\, \frac{k}{2}\,\theta \leqslant (\frac{k\theta}{2})^{2\ell}$, and hence the right-hand side of (4.5) follows with $C_2 = \frac{1}{\binom{2\ell}{\ell}}$. As $\alpha(k, 2v, \lambda) \geqslant 0$,

$$1 - a_\ell(k,\theta) \geqslant \frac{4^\ell}{\binom{2\ell}{\ell}} \sum_{v=0}^{[k/4]} \alpha(k, 2v, \lambda) \sin^{2\ell}\, \frac{k-2v}{2}\, \theta.$$

Using $\sum_{v=0}^{[k/4]} \alpha(k, 2v, \lambda) > \beta > 0$, and as for $v < [\frac{k}{4}]$, $\sin^{2\ell}\, \frac{k-2v}{2}\,\theta \geqslant \sin^{2\ell}\, \frac{k}{4}\,\theta \geqslant (\frac{k\theta}{2\pi})^{2\ell}$, we have the estimate $C_1 \geqslant \frac{\beta}{\binom{2\ell}{\ell}}\, \frac{1}{\pi^{2\ell}} > 0$. To obtain (4.6) for $0 < \tau \leqslant \pi$ and $k\theta \leqslant \pi$ we use the lower estimate of (4.5) and obtain $1 - a_\ell(k,\theta) \geqslant C_1\tau^{2\ell}$ or $a_\ell(k,\theta) \leqslant 1 - C_1\tau^{2\ell}$, and we may set $v_{d,\ell,\tau} = 1 - C_1\tau^{2\ell} < 1$ for $0 < \tau \leqslant k\theta$. For $k\theta \geqslant \pi$ (regardless of τ) we set

$$1 - a_\ell(k,\theta) \geqslant \frac{4^\ell}{\binom{2\ell}{\ell}} \sum_{v\in I(k)} \left(\sin\, \frac{k-2v}{2}\,\theta\right)^{2\ell} \alpha(k, 2v, \lambda),$$

where $I(k,\theta)=\bigcup_{m=0}^{[\frac{k\theta}{2\pi}-\frac{1}{2}]}\{v: 0\leqslant v\leqslant [\frac{k}{2}],\ \frac{\pi}{4}+m\pi\leqslant (k-2v)\frac{\theta}{2}\leqslant\frac{3\pi}{4}+m\pi\}$, and obtain

$$1-a_\ell(k,\theta)\geqslant\frac{2^\ell}{\binom{2\ell}{\ell}}\sum_{v\in I(k)}\alpha(k,2v,\lambda).$$

Using (4.8), we have $\alpha(k,2v,\lambda)\geqslant\frac{A}{k}$ with $A=A(\lambda)>0$ where $A(\lambda)$ is independent of k. As the number of elements in $I(k)$ is greater than Bk with $B>0$ for $k\geqslant k_0$ ($k_0=10$ say), (4.6) follows for $k\geqslant k_0$. For $1\leqslant k<k_0$ (4.6) follows directly from (4.9) (recall $\theta\in[0,\frac{\pi}{2}]$). □

Remark 4.5. Since for $L_2(S^{d-1})$ the realization

$$K_\ell(f,\Delta,n^{-2\ell})_2\approx\|f-S_nf\|_2+n^{-2\ell}\|\widetilde{\Delta}^{2\ell}S_nf\|_2$$

with S_n the L_2 projection on span $\bigcup_{k=0}^{n}H_k$ holds, Lemma 4.4 yields Theorem 4.1 for $p=2$. For $p\neq 2$ we still need some work.

The following lemma (or variants thereof) was used earlier (see for instance [Da]). We state the present variant for the convenience of the reader. Recall $\Delta m_k=m_{k+1}-m_k$, $\Delta^j m_k=\Delta(\Delta^{j-1}m_k)$ and $\Delta^0 m_k=m_k$.

Lemma 4.6. (a) *For sequences a_k and b_k we have*

$$\Delta^j(a_kb_k)=\sum_{s=0}^{j}\binom{j}{s}(\Delta^{j-s}a_k)(\Delta^s b_{k+j-s}). \tag{4.10}$$

(b) *For a sequence A_k satisfying $A_k\geqslant A>0$*

$$\begin{aligned}|\Delta^jA_k^{-1}|&\leqslant\frac{1}{|A_k|}\sum_{s=0}^{j-1}\binom{j}{s}|\Delta^sA_k^{-1}|\,|A^{j-s}A_{k+s}|\\&\leqslant C\max_{0\leqslant s\leqslant j}|\Delta^sA_k^{-1}|\,|\Delta^{j+s}A_{k+s}|\end{aligned}$$

with $C=\frac{1}{A}2^j$.

Proof. We obtain the identity (4.10) for $j=0,1$ by inspection. For higher j one proves (4.10) by mathematical induction. Part (b) follows from the observation $A_k^{-1}A_k=1$, choosing $A_k^{-1}=a_k$, $A_k=b_k$ in (4.10) and using $A_k\geqslant A>0$. □

Perhaps the crucial estimate needed for the proof of Theorem 4.1 is given in the following lemma.

Lemma 4.7. *Suppose $\theta \in [0, \frac{\pi}{2}]$, $a_\ell(k,\theta)$ is given by* (4.3) *and $\lambda = \frac{d-2}{2}$. Then for any integer j, $j \geqslant 1$, and any $\tau > 0$ such that for $0 < k\theta < \tau$ we have*

$$\left|\Delta^j \frac{1 - a_\ell(k,\theta)}{(k(k+2\lambda)\theta^2)^\ell}\right| \leqslant C_{\ell,\tau,j}(k^{-j+1}\theta + k^{-j-1}). \tag{4.11}$$

Proof. We set $f_k(t) = Q_k^{(\lambda)}(\cos t)$, and using (4.3), we have

$$1 - a_\ell(k,\theta) = \frac{(-1)^\ell}{\binom{2\ell}{\ell}} \int_{-\theta/2}^{\theta/2} \cdots \int_{-\theta/2}^{\theta/2} f_k^{(2\ell)}(u_1 + \cdots + u_{2\ell})\, du_1 \cdots du_{2\ell} \tag{4.12}$$

as $Q_k^{(\lambda)}(\cos t) = Q_k^{(\ell)}\big(\cos(-t)\big)$. We now set $g_k(x) = Q_k^{(\lambda)}(x)$ and write for $k \geqslant 2\ell$

$$f_k^{(2\ell)}(t) = \sum_{s=1}^{2\ell} g_k^{(s)}(\cos t) \sum_{\max(s-\ell,0) \leqslant i \leqslant [\frac{s}{2}]} C(s,\ell,i)(\sin t)^{2i}(\cos t)^{s-2i}. \tag{4.13}$$

Recall now, using [Sz, (4.7.3) and (4.7.14)], that for $\mu > 0$

$$\frac{d}{dx} Q_k^{(\mu)}(x) = \frac{k(k+2\mu)}{2\mu+1} Q_{k-1}^{(\mu+1)}(x),$$

from which we may deduce

$$g_k^{(s)}(x) = \left(\frac{d}{dx}\right)^s Q_k^{(\lambda)}(x) = C_s(\lambda)\varphi_s(k) Q_{k-s}^{(\lambda+s)}(x), \tag{4.14}$$

where $C_s(\lambda) = (2\lambda+1)\cdots(2\lambda+2s-1)$ and $\varphi_s(k)$ is a polynomial in k of degree $2s$. Using (4.12) and (4.13), it is sufficient to show that for $2\ell \leqslant k$, $kt \leqslant k\theta < \ell\tau$, $j \geqslant 1$

$$(\sin t)^\delta \left|\Delta^j \frac{g_k^{(s)}(\cos t)}{\big(k(k+2\lambda)\big)^\ell}\right| \leqslant C(k^{-j+1}t + k^{-j-1}) \quad \text{with} \quad \delta = \begin{cases} 0, & s \leqslant \ell, \\ 2(s-\ell), & s > \ell. \end{cases}$$

Using (4.10) with $a_k = \frac{\varphi_s(k)}{\big(k(k+2\lambda)\big)^\ell}$ and $b_k = Q_{k-s}^{(\lambda+s)}(\cos t)$, observing that

$$\left|\Delta^\nu \left(\frac{\varphi_s(k)}{k(k+\lambda)^\ell}\right)\right| \leqslant \begin{cases} Ck^{2s-2\ell-\nu} & \text{if } s \neq \ell \text{ or } s = \ell \text{ and } \nu = 0, \\ Ck^{-\nu-1} & \text{if } s = \ell \text{ and } \nu > 0 \end{cases} \tag{4.15}$$

and following Lemma 3.2 of Belinsky et al. [Be-Da-Di] which implies

$$|\Delta^\mu Q_{k-s}^{(\lambda+s)}(\cos t)| \leqslant C_1 t^\mu, \tag{4.16}$$

we recall $tk \leqslant \theta k \leqslant \tau\ell$ to obtain for $s > \ell$

$$\begin{aligned}(\sin t)^{2(s-\ell)} \left|\Delta^j \frac{g_k^{(s)}(\cos t)}{\big(k(k+2\lambda)\big)^\ell}\right| &\leqslant C_2 \max_{\substack{0 \leqslant \nu \leqslant j \\ \nu \in \mathbb{Z}_+}} t^{2(s-\ell)} k^{2s-2\ell-\nu} t^{j-\nu} \\ &\leqslant C_3 k^{-j+1} t.\end{aligned}$$

For $s \leqslant \ell$ we use (4.15) and (4.16) to derive

$$\left| \Delta^j \frac{g_k^{(s)}(\cos t)}{\big(k(k+2\lambda)\big)^\ell} \right| \leqslant C\Big(\max_{\substack{0\leqslant v\leqslant j \\ v\in\mathbb{Z}_+}} k^{-v-1} t^{j-v} + t^j \Big) \leqslant C_1\big(k^{-j-1} + k^{-j+1} t\big).$$

This concludes the proof for $k \geqslant 2\ell$. We note that for $1 \leqslant k \leqslant 2\ell$ (4.11) is obvious as $\big(1 - a_\ell(k,\theta)\big)/\big(k(k+2\lambda)\theta^2\big)^\ell$ is bounded. (In any case the lemma is needed only for $k \geqslant k_0$ for some fixed k_0.) $\square$

5. The proof of Theorem 4.1

We first state the realization result which will be used.

We define the operator $\eta_{a\theta}(f)$ using the function $\eta(x)$ satisfying $\eta(x) \in C^\infty(R_+)$, $\eta(x) = 1$ for $0 \leqslant x \leqslant 1$, and $\eta(x) = 0$ for $x \geqslant 2$. The operator $\eta_{a\theta}(f)$ is given by

$$\eta_{a\theta}(f) = \sum_{k=0}^{\infty} \eta(a\theta k) P_k(f) \tag{5.1}$$

where

$$f \sim \sum_{k=0}^{\infty} P_k(f).$$

Following [Ch-Di,Di], one can obtain the realization theorem by $\eta_{a\theta}(f)$, which is a De la Vallée Poussin-type operator.

Realization Theorem. *For $f \in L_p(S^{d-1})$ and any positive a*

$$K_\ell(f, \widetilde{\Delta}, \theta^{2\ell})_p \approx \|f - \eta_{a\theta}(f)\|_p + \theta^{2\ell} \|\widetilde{\Delta}^\ell \eta_{a\theta}(f)\|_p, \tag{5.2}$$

where $K_\ell(f, \widetilde{\Delta}, \theta^{2\ell})_p$ is given in (4.1) *and $\widetilde{\Delta}$* is the Laplace–Beltrami operator.

The above theorem has a somewhat different statement than in [Ch-Di, Theorem 4.5] or [Di, Theorem 7.1] but the proof, and in fact the theorem itself, is the same.

Proof of Theorem 4.1. Following the proof of Theorem 3.1 and the realization result in this section, we have to show for some positive a

$$\|f - S_{\ell,\theta}(f)\|_p \geqslant C_1 \|f - \eta_{a\theta}(f)\|_p, \tag{5.3}$$

$$\|f - S_{\ell,\theta}(f)\|_p \geqslant C_2 \theta^{2\ell} \|\widetilde{\Delta}^\ell \eta_{a\theta}(f)\|_p \tag{5.4}$$

and

$$\|\eta_{a\theta}(f) - S_{\ell,\theta}\big(\eta_{a\theta}(f)\big)\|_p \leqslant C_3 \theta^{2\ell} \|\widetilde{\Delta}^\ell \eta_{a\theta}(f)\|_p. \tag{5.5}$$

To prove (5.3) it is sufficient to show

$$\|f-\eta_{a\theta}(f)-(I+S_{\theta,\ell}+\ldots+S^4_{\theta,\ell})(I-\eta_{a\theta})\big(f-S_{\theta,\ell}(f)\big)\|_p \\ \leqslant C_4(\ell,p)\|f-S_{\theta,\ell}f\|_p \tag{5.6}$$

as

$$\|(I+S_{\theta,\ell}+\cdots+S^4_{\theta,\ell})(I-\eta_{a\theta})\big(f-S_{\theta,\ell}(f)\big)\|_p \leqslant C_5\|f-S_{\theta,\ell}(f)\|_p$$

since $S_{\theta,\ell}$ is a bounded operator.

To prove (5.6) we have to show that

$$\mu_\ell(k,\theta)=\big(1-\eta(a\theta k)\big)\,\frac{a_\ell(k,\theta)^5}{1-a_\ell(k,\theta)}$$

is a multiplier operator on $f\in L_p(S^{d-1})$. We note that for $k\leqslant\frac{1}{a\theta}$, $\mu_\ell(k,\theta)=0$. We now recall that as the Cesàro summability of order m with $m>\frac{d-2}{2}$ is a bounded operator in $L_p(S^{d-1})$, $1\leqslant p\leqslant\infty$, (see [Bo-Cl]), the condition for $\mu(k)$ to be a bounded multiplier operator is (see [Ch-Di] or [Be-Da-Di] or numerous other places)

$$\sum_{k=0}^{\infty}|\Delta^{m+1}\mu(k)|\binom{k+m}{m}<M.$$

For $\mu(k)=\mu_\ell(k,\theta)$ we note that for $k\geqslant\frac{1}{a\theta}$ i.e. $k\theta\geqslant\frac{1}{a}$ (4.6) implies

$$1-a_\ell(k,\theta)\geqslant 1-v_{d,\ell}>0.$$

Therefore, using Lemma 4.3, we have for $k\theta\geqslant\frac{1}{a}$

$$|\Delta^j\mu_\ell(k,\theta)|\leqslant C_6\theta^j\left(\frac{1}{k\theta}\right)^{5\lambda}.$$

We choose $m=[\frac{d}{2}]>\frac{d-2}{2}$, $j=m+1$, $\lambda=\frac{d-2}{2}$ and as $\binom{k+m}{m}\leqslant Ak^m$, we have

$$\left|\binom{k+m}{m}\Delta^{m+1}\mu_\ell(k,\theta)\right|\leqslant C_7\theta^{[d/2]+1}k^{[d/2]}\left(\frac{1}{k\theta}\right)^{5(\frac{d}{2}-1)} \\ \leqslant C_7\theta^{[d/2]-5\,\frac{d}{2}+6}k^{[d/2]-5\,\frac{d}{2}+5}.$$

Using $[d/2]-5\,\frac{d}{2}+5<-1$ for $d\geqslant 3$ and $\mu_\ell(k,\theta)=0$ for $k\leqslant\frac{1}{a\theta}$, we have

$$\sum\binom{k+[d/2]}{[d/2]}\left|\Delta^{[d/2]+1}\mu_\ell(k,\theta)\right|\leqslant M.$$

To prove (5.5) we have to show that

$$\mu_\ell(k,\theta)=\frac{1-a_\ell(k,\theta)}{\big(k(k+2\lambda)\theta^2\big)^\ell}\,\eta(a\theta k)$$

is a multiplier, or as $\mu_\ell(k,\theta)$ are finite, that for $m=[\frac{d}{2}]$

$$\sum_{k=k_0}^{\infty} |\varDelta^{m+1}\mu_\ell(k,\theta)|k^m = \sum_{k=k_0}^{[\frac{2}{a\theta}+m+1]} |\varDelta^{m+1}\mu_\ell(k,\theta)|k^m < M \tag{5.7}$$

with M independent of θ. Using Lemmas 4.7 and 4.6(a) with $a_k = \frac{1-a_\ell(k,\theta)}{\left(k(k+2\lambda)\theta^2\right)^\ell}$ and $b_k = \eta(a\theta k)$, we derive (5.7) as $|\varDelta^r b_k| \leqslant C(a\theta)^r$. To prove (5.4) we have to show that

$$\mu'_\ell(k,\theta) = \frac{\left(k(k+2\lambda)\theta^2\right)^\ell}{1-a_\ell(k,\theta)}\,\eta(a\theta k)$$

also satisfies (5.7). We now use Lemmas 4.7 and 4.6(a) and (b) and replace a_k above by a_k^{-1}. We note that Lemma 4.6(b) is applicable as $a_k = \frac{1-a_\ell(k,\theta)}{\left(k(k+2\lambda)\theta^2\right)^\ell} \geqslant A > 0$ by (4.5). We further note that as $a_k^{-1} \geqslant C_1^{-1} > 0$ with C_1 of (4.5), (b) of Lemma 4.6 and mathematical induction imply that (4.11), which was proved for a_k, is valid for a_k^{-1} as well. $\square$

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